PROBLEM OF THE DAY

In a magic square each row, each column, and the two main diagonals have the same totals.

Complete the partially filled in magic square shown.

	1	
5		
7		3

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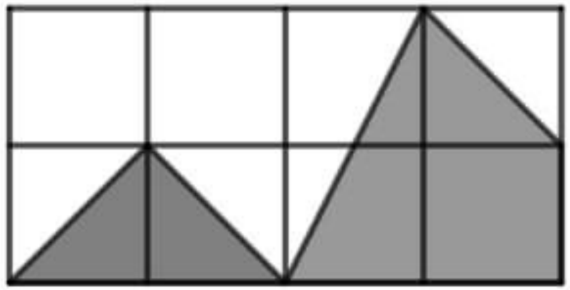
15	1	11
5	9	13
7	17	3

PROBLEM OF THE DAY

The rectangle is divided into eight equal squares

What fraction of the area of the square is shaded?



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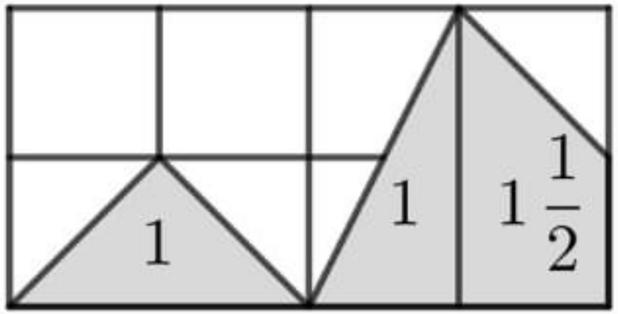
SOLUTION 2

$$\text{Fraction shaded} = \frac{1+1+1\frac{1}{2}}{8}$$

$$= \frac{3\frac{1}{2}}{8} = \frac{7}{16}$$



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PROBLEM OF THE DAY 3

Three apples, two oranges and one cabbage cost £2.29.
Four apples, three oranges and two cabbages cost £3.77.

How much do five apples, four oranges and three cabbages cost?

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SOLUTION 3

Because three apples, two oranges and one cabbage cost £2.29, and four apples, three oranges and two cabbages cost £3.77, we deduce that the cost of one apple, one orange and one cabbage is

$$£3.77 - £2.29 = £1.48.$$

Hence five apples, four oranges and three cabbages cost

$$£3.77 + £1.48 = £5.25.$$

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PROBLEM OF THE DAY 4

A manufacturer sells *Super Soup* in cylindrical cans of height 10 cm and radius 4 cm.

They rebrand the soup as *Square Meal Soup* in cans containing the same volume of soup as before, but looking square because they have the same diameter as their height.

What is the height of the new cans?

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SOLUTION 4

The volume of the original can, in cm^3 is given by

$\pi(4^2) \times 10 = 160\pi$.

Let the height of the new can be h cm.

Then it also has diameter h cm and hence radius $h/2$ cm.

Therefore the volume of the new can, in cm^3, is given by

$\pi(h/2)^2 \times h = \pi h^3/4$.

Therefore $\pi h^3/4 = 160\pi$. It follows that $h^3 = 640$.

Hence $h = \sqrt[3]{640} = 4\sqrt[3]{10} \approx 8.62$.

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PROBLEM OF THE DAY 5

Alice said "All four of us are telling the truth."
The Cheshire Cat said "None of us is telling the truth."
The Dormouse said "Two or three of us are telling the truth."
The Gryphon said "I am the only one of us who is telling the truth."

Who is telling the truth?

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SOLUTION 5

The four statements contradict each other, so at most one of them is true. If the Cheshire Cat's statement were true, not all four of them would not be telling the truth. So the Cheshire Cat's statement is not true. Therefore exactly one of them is telling the truth. We deduce that it is the Gryphon who is tellng the truth.

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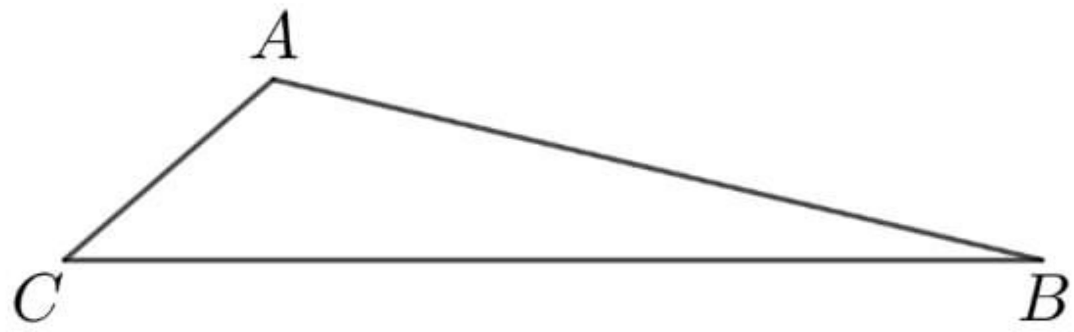
PROBLEM OF THE DAY 6



In this triangle $\angle BCA$ is twice $\angle ABC$, and $\angle BAC$ is three times $\angle BCA$.

What are the three angles of the triangle?

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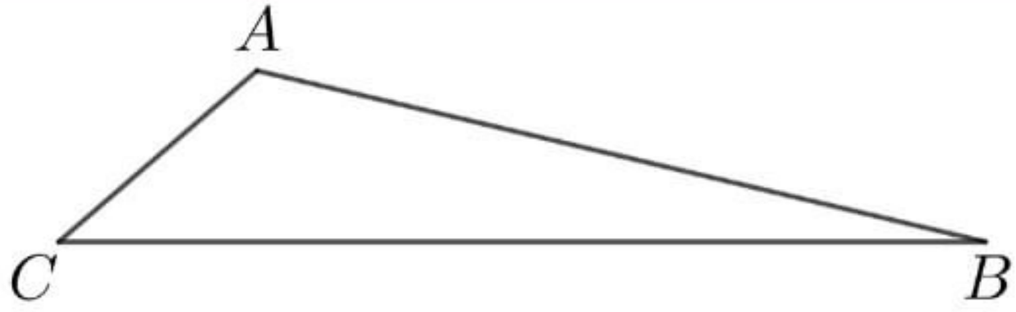
SOLUTION 6



Suppose $\angle ABC = x^\circ$.
Then $\angle BCA = 2x^\circ$ and $\angle ABC = 6x^\circ$.

The sum of the angles in a triangle is 180°.
Therefore $x + 2x + 6x = 180$.
Hence $9x = 180$ and therefore $x = 20$.
It follows that the angles in the triangle are 20°, 40° and 120°.

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PROBLEM OF THE DAY 7

Which is the least positive integer which when multiplied by 450 gives an answer that is a cube?

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SOLUTION 7

The prime factorization of 450 is

$$450 = 2 \times 3^2 \times 5^2.$$

In the prime factorization of a cube the exponent of each prime is a multiple of 3.

It follows that the smallest multiple of 450 that is a cube is

$$2^3 \times 3^3 \times 5^3.$$

This number is obtained when 450 is multiplied by

$$2^2 \times 3 \times 5 = 60.$$

Therefore 60 is the required answer.

$[450 \times 60 = 27000 = 30^3.]$

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PROBLEM OF THE DAY 8

I bought some first class stamps at 70p each and some second class stamps at 61p each. I spent a total of £11.

How many stamps did I buy?

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SOLUTION 8

Let f be the number of first class stamps and s be the number of second class stamps that I bought. Then, because I spent £11 on the stamps,

$$70f + 61s = 1100. \quad (1)$$

This is one equation involving two unknowns, but in this problem f and s are positive integers. We will see that it follows that the equation has just one solution.

Equation (1) may be rearranged as

$$61s = 1100 - 70f = 10(110 - 7f).$$

It follows that s is a multiple of 10. Now, by (1), $61s \leq 1200$ and hence $s < 20$. Therefore $s = 10$.

Hence, from (1), $70f = 1100 - 10 \times 61 = 1100 - 610 = 490$.

Therefore $f = 7$.

It follows that I bought 17 stamps, 7 first class and 10 second class.

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PROBLEM OF THE DAY 9

Exactly one of these integers is a prime number.

A	45
B	456
C	4567
D	45678
E	456789

Which is it?

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SOLUTION 9

We are told that exactly one of the numbers is prime. We can identify it by eliminating the numbers that are not prime.

We see that 45 is a multiple of 5. Also 456 and 45678 are both multiples of 2. So none of these three numbers is prime.
This leave 4567 and 456789.

Now

$$4 + 5 + 6 + 7 + 8 + 9 = 39 = 3 \times 13.$$

Because the sum of the digits of 456789 is a multiple of 3, it follows that 456789 is a multiple of 3 and hence is not prime.

This leaves 4567 as the only prime number in the list.

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PROBLEM OF THE DAY 10

Each of a transport company's lorries can carry a load of up to 10 tons.
The company has 40 crates each weighing 3 tons to transport.

How many lorry loads is that?

SOLUTION 10

Each lorry can carry 3 crates but no more.
Therefore to move 40 crates requires 14 lorry loads .

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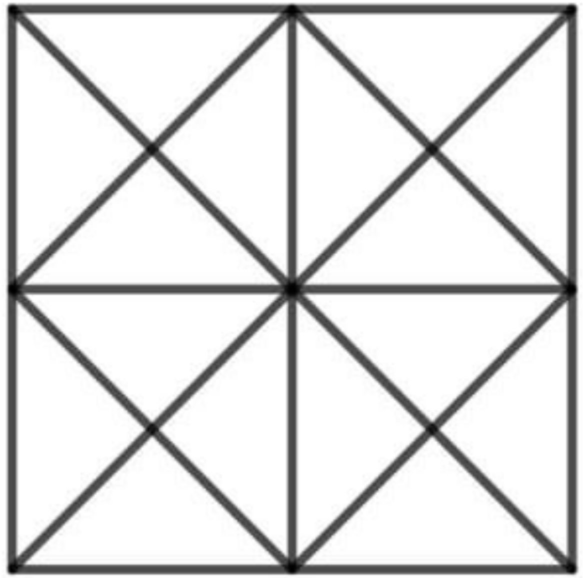
PROBLEM OF THE DAY 11



How many triangles of any size are there in this diagram?

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SOLUTION 11

There are 16 small triangles, one of which is shown here in green; 16 triangles (blue) made up of two small triangles; 8 triangles (orange) made up of four small triangles; and 4 triangles (red) made up of eight small triangles.

This makes a total of $16+16+8+4 = 44$ triangles.

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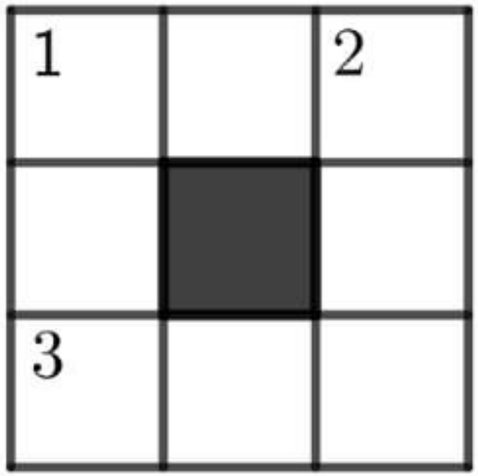
PROBLEM OF THE DAY 12



Complete this crossnumber using eight different non-zero digits.

Across	*Down*
1. A square	1. A prime
3. A cube	2. A prime

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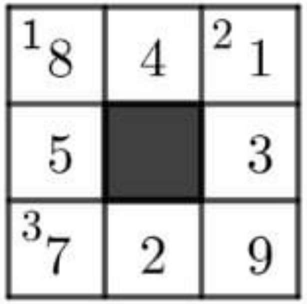
SOLUTION 12



The 3-digit cubes are 125, 216, 343, 512 and 729.
2 Down is a prime so its units digit is 1, 3, 7 or 9.
3 Across cannot repeat digits. It follows that 3 Across is 729.

The only 3-digit squares that do not use the digits of 729 and which do not repeat digits are 361 and 841.

It can be checked that the crossnumber cannot be completed if 1 Across is 361, and if 1 Across is 841, the only way to use the remaining non-zero digits so that that 1 Down and 2 Down are both primes, is as above.

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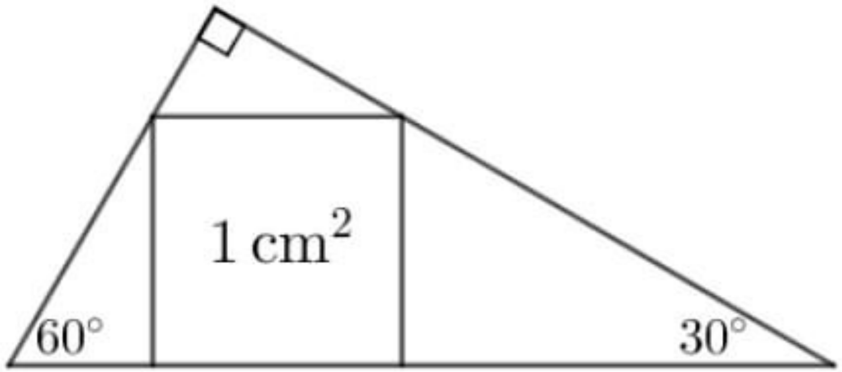
PROBLEM OF THE DAY 13



The diagram shows a square with area $1\,\text{cm}^2$ inside a right-angled triangle whose other two angles are 30° and 60°.

Two vertices of the square lie on the hypotenuse of the triangle, with one vertex on each of the other two sides.

What is the area of the triangle?

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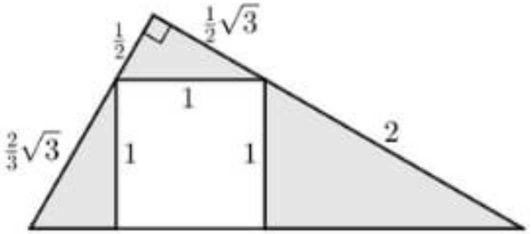
SOLUTION 13



The three shaded triangles are similar to the large triangle.
That is, their angles are also 90°, 60° and 30°.
It follows the lengths of their hypotenuses, longest other sides and shortest sides are in ratio $2 : \sqrt{3} : 1$.

Therefore their side lengths, in centimetres, are as shown in the diagram.
Hence the sides of the large triangle adjacent to the right angle have lengths $(\frac{1}{2} + \frac{2}{3}\sqrt{3})$ cm and $(2+\frac{1}{2}\sqrt{3})$ cm.

From the formula area $= \frac{1}{2}$(base $\times$ height), we can now deduce that the area of the large triangle is

$$\tfrac{1}{2}(\tfrac{1}{2}+ \tfrac{2}{3}\sqrt{3})(2+ \tfrac{1}{2}\sqrt{3})\ \text{cm}^2.$$

This works out to be

$$(1 + \tfrac{19}{24}\sqrt{3})\ \text{cm}^2.$$

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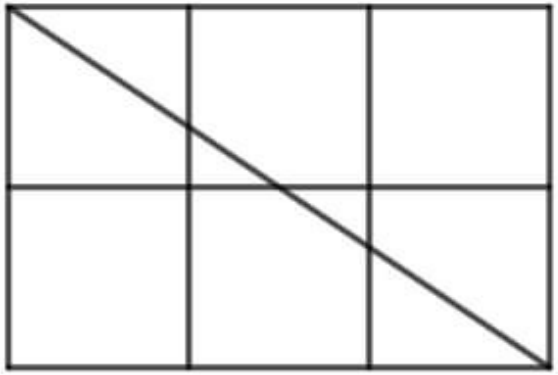
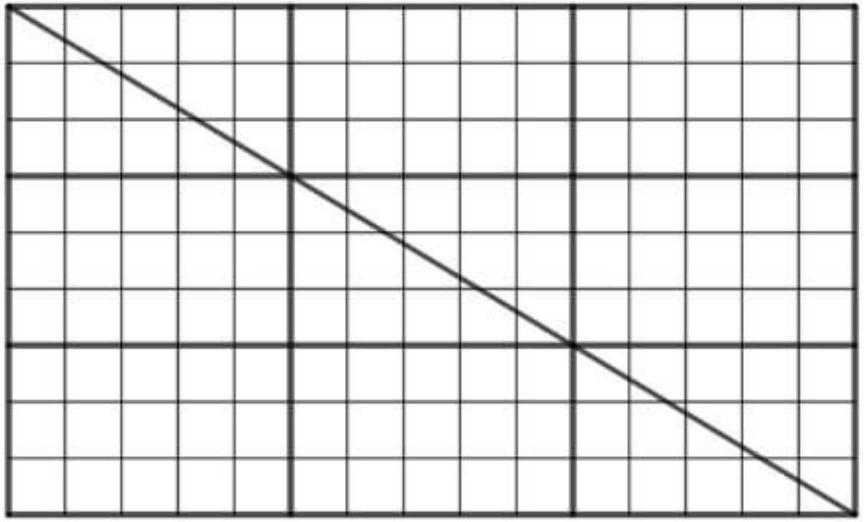
PROBLEM OF THE DAY 14



The diagram shows that a diagonal of a 2×3 grid of squares, passes through 4 of the squares.

In a 9×15 grid of squares, how many squares does the diagonal pass through?

SOLUTION 14



In the diagram the 9×15 grid has been divided into nine 3×5 grids.

It may be seen that the diagonal shown goes through 3 of these 3×5 grids, and that passes through 7 squares in each of them.

Therefore, in total the diagonal passes through $3 \times 7 = 21$ squares.

Investigate some other grids, for example a 5×7 and a 6×8 grid.

Can you find a formula, in terms of m and n for the number of squares that the diagonal of an $m \times n$ grid passes through?

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PROBLEM OF THE DAY 15

+ − × ÷

$$9 \div (8 - 7) + 6 - 5 \times (4 - 3 + 2) + 1 = 1.$$

Can you make

$$2020$$

using just the digits 9, 8, 7, 6, 5, 4, 3, 2, 1
in this order and the symbols $+ \ - \ \times \ \div \ (\ )$?

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SOLUTION 15

$$9 \times 8 \times 7 \times (6 - 5) \times 4 + 3 + 2 - 1 = 2020$$

Other solutions are possible.

Now try making other numbers in this way.

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PROBLEM OF THE DAY 16

Which number word can be inserted in the gap in the following sentence to make it true?

> It may be seen that the letter "e" occurs .... times in this sentence.

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SOLUTION 16

thirteen

It may be seen that the letter "e" occurs *thirteen* times in this sentence.

Note : This is not a very precise question. If foreign words are allowed, then *thirteen* could, for example, be replaced by the French word *douce*["twelve"] or the German *dreizehn* ["thirteen"] Can you find any more?

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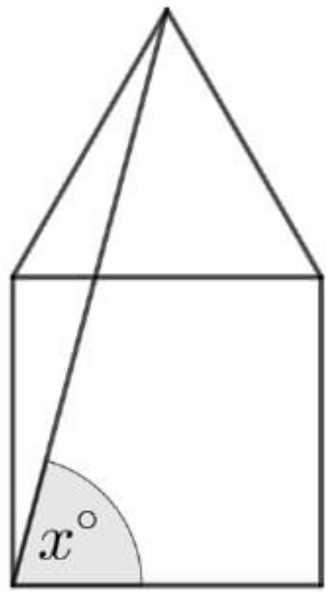
PROBLEM OF THE DAY 17



A square and an equilateral triangle share an edge, with an additional line, as shown in the diagram.

Find the angle x°, and find $\tan x^\circ$ in the form $a + \sqrt{b}$, where a and b are integers.

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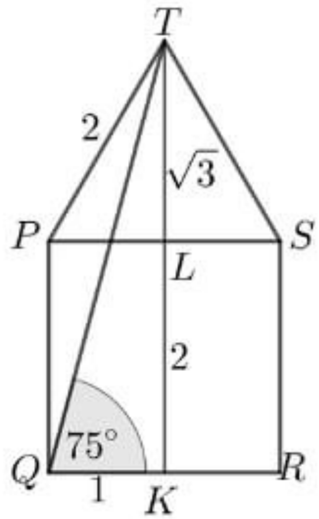
SOLUTION 17

$\boldsymbol{x^\circ = 75^\circ.}\quad \boldsymbol{\tan 75^\circ = 2 + \sqrt{3}.}$



We have $PT = PS = PQ$. Therefore the triangle TPQ is isosceles and so $\angle PQT = \angle PTQ$. $\angle TPQ = 60^\circ + 90^\circ = 150^\circ$. It follows that $\angle PQT = \frac{1}{2}(180^\circ - 150^\circ) = 15^\circ$.
Hence $\angle TQK = 90^\circ - 15^\circ = 75^\circ$.

We suppose the side lengths of the square and the triangle are 2.

It follows that $\tan 75^\circ = \dfrac{TK}{QK} = \dfrac{2+\sqrt{3}}{1} = 2+\sqrt{3}$.

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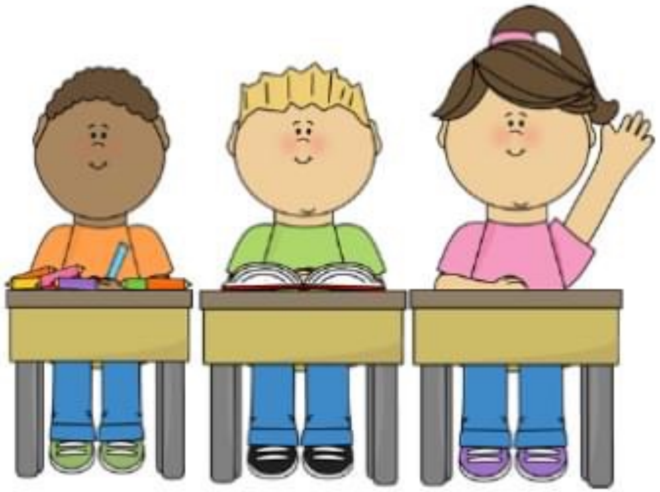
PROBLEM OF THE DAY 18



At the Upper Kingswood Modern Technical school, there are 142 A-level students.

Of these students, 65 are studying Mathematics, 38 Physics, and 49 History. There are 27 students studying Mathematics and Physics, but not History; 4 studying History and Physics, but not Mathematics; and 12 studying History and Mathematics, but not Physics.

43 students study none of these subjects.

How many students are studying Mathematics, Physics and History?

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SOLUTION 18

5

Note : We give a fairly sophisticated solution. Other methods are possible.

We let M, H and P be the sets of students studying Mathematics, History and Physics, respectively.

We use the notation $\#(X)$ for the number of students in the set X. In this notation the number we are asked to find is $\#(M \cap H \cap P)$.

By the *Inclusion-Exclusion Principle*,
$\#(M \cup H \cup P) =$
$[\#(M) + \#(H) + \#(P)] - [\#(M \cap H) + \#(M \cap P) + \#(H \cap P)] + \#(M \cap H \cap P)$.

Here $\#(M \cup H \cup P)$ is the number of students studying at least one of Mathematics, History and Physics, and hence is $142 - 43 = 99$.
Also, $\#(M) + \#(H) + \#(P) = 65 + 49 + 38 = 152$.

Now suppose there are x students studying Mathematics, History and Physics. Because there are 12 students studying Mathematics and History, but not Physics, there are $12 + x$ students altogether studying Mathematics and History. That is $\#(M \cap H) = 12 + x$. Similarly $\#(M \cap P) = 27 + x$, and $\#(H \cap P) = 4 + x$.

Hence, by the above formula, $99 = 152 - [(12 + x) + (27 + x) + (4 + x)] + x$.
This gives $99 = 152 - (43 + 3x) + x$, and hence $2x = 152 - 43 - 99 = 10$.
Therefore $x = 5$.

Therefore there are 5 students who study Mathematics, History and Physics.

PROBLEM OF THE DAY 19

A coin is biased so that the probability of getting a head is 0.6.

What is the probability of getting exactly three heads if the coin is tossed five times?

What is the probabilty of getting at least three heads?

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SOLUTION 19

$$\mathbf{\frac{216}{625} = 0.3456; \quad \frac{2133}{3125} = 0.68256}$$

The probability of a head is 0.6. Therefore the probablity of a tail is 0.4. Hence the probability of a particular sequence of 3 heads and 2 tails, such as HHTHT, is $0.6 \times 0.6 \times 0.4 \times 0.6 \times 0.4 = 0.6^3 \times 0.4^2$.

There are 10 ways in which a sequence of 5 tosses can result in 3 heads and 2 tails. Therefore the probability of getting exactly 3 heads is $10 \times 0.6^3 \times 0.4^2 = \frac{216}{625} = 0.3456$.

In a similar way, the probability of exactly 4 heads is $5 \times 0.6^4 \times 0.4^1 = 0 = \frac{162}{625} = 0.2592$ and the probability of 5 heads is $1 \times 0.6^5 = \frac{243}{3125} = 0.07776$.

Therefore the probability of getting at least 3 heads is $\frac{216}{635} + \frac{162}{625} + \frac{243}{3125} = \frac{2133}{3125} = 0.68256$.

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PROBLEM OF THE DAY 20

1111....1111

A *repunit* (short for "repeated units") is a positive integer which may be written as a sequence of 1s.

The first four repunits are 1, 11, 111 and 1111.

Which is the smallest repunit that is a multiple of 99?

More generally: which positive integers can be a factor of a repunit?

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SOLUTION 20

111 111 111 111 111 111

$99 = 9 \times 11$. It follows that to be a multiple of 99 a positive integer needs to be multiple of both 9 and 11.

The test for whether an integer is a multiple of 9 is that the sum of its digits is a multiple of 9. Therefore the repunits that are multiples of 9 are those in in which the number of 1s is a multiple of 9.

A repunit is a multiple of 11 if it is made up of an even number of 1s.

Therefore the repunits that are multiples of 99 are those made up of a number of 1s that is both even and a multiple of 9. The smallest such repunit is

$$111\,111\,111\,111\,111\,111$$

containing eighteen 1s.

[Note that $111\,111\,111\,111\,111\,111 = 99 \times 1122334455667789$.]

It can be proved that each integer that is divisible neither by 2 nor by 5 is a factor of infinitely many repunits.

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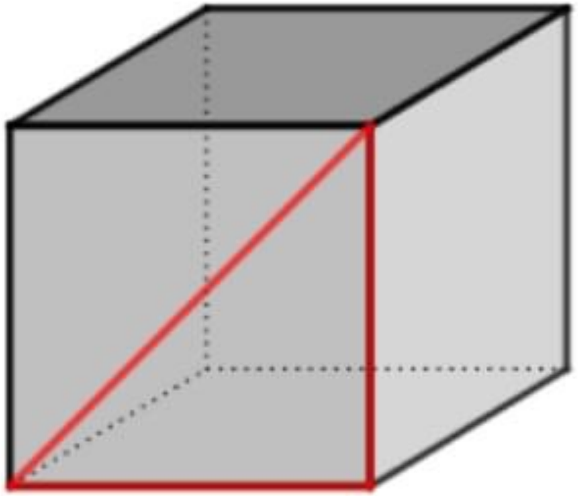
PROBLEM OF THE DAY 21



The diagram shows a right-angled triangle whose vertices are three vertices of the cube.

How many right-angled triangles are there altogether whose vertices are three vertices of the cube?

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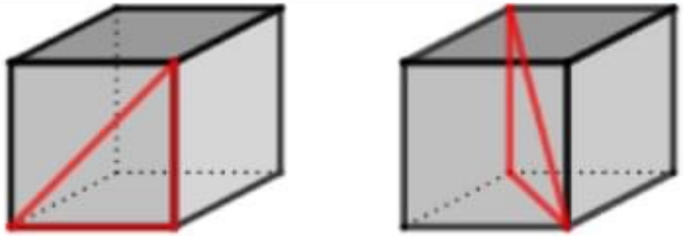
SOLUTION 21



There are two sorts of right-angled triangles whose vertices are three vertices of the cube.

On each of the six faces there are four triangles similar to that shown on the left, which uses three of the vertices on that face. That makes $6 \times 4 = 24$ triangles of this type.

There are also four triangles of the kind shown on the right. For each of the six pairs of diagonally opposite edges, there are four triangles that use three of the vertices on these edges. That makes another $6 \times 4 = 24$ triangles.

This gives a total of $24 + 24 = 48$ triangles.

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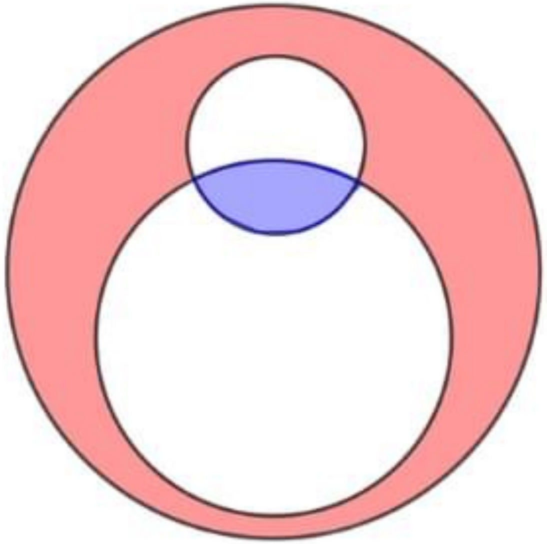
PROBLEM OF THE DAY 22

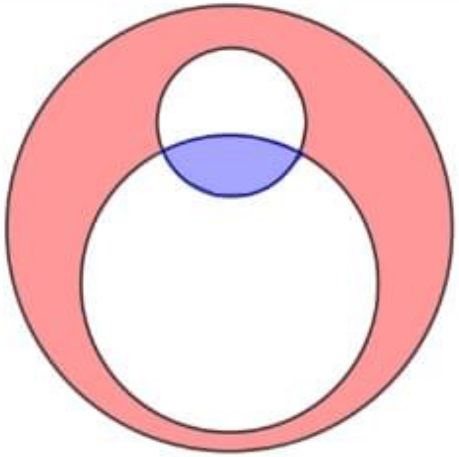


The diagram shows overlapping circles with radii 1 cm and 2 cm inside a circle of radius 3 cm.

The area of the region shown in red is half the area of the largest circle.

What is the area of the region, shown in blue, where the two smaller circles overlap?

SOLUTION 22



Let the area, in cm^2, of the region where the two smaller circles overlap be x.
We let R be the region made up of all points which are in one or both of the smaller circles.
The area of the largest circle, in cm^2, is $\pi(3^2) = 9\pi$.
The area of the region inside the larger circle but outside the two smaller circles is a half of this. Hence so also is the area of the region R. Therefore the area of R, in cm^2, is $\frac{9}{2}\pi$.

On the other hand, the area of R is the sum of the areas of the two smaller circles, less the area of the region where they overlap. Therefore

$$\tfrac{9}{2}\pi = \pi(1^2) + \pi(2^2) - x,$$

and hence $x = \pi + 4\pi - \frac{9}{2}\pi = \frac{1}{2}\pi$. Therefore the area of the overlap is $\frac{1}{2}\pi$ cm^2.

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PROBLEM OF THE DAY 23

$$3\ 7\ \_\ \_\ 6\ 4\ \_$$

Three of the digits are missing in the seven-digit number shown above.

This number is divisible by 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12.

Find the seven-digit number.

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SOLUTION 23

3797640

Let n be the number we seek. Let the missing digits be a, b and c, so that $n =$ "37ab64c" Because n is a multiple of 10, we have $c = 0$ and hence $n =$ "37ab640".

Because n is a multiple of 9, the sum of its digits is a multiple of 9. Therefore, $3 + 7 + a + b + 6 + 4 + 0 = a + b + 20$ is multiple of 9.

Because a and b are digits, $0 \leq a + b \leq 18$. Hence $20 \leq a + b + 20$ ≤ 38. Therefore $a + b + 20$ is either 27 or 36, and so $a + b$ is either 7 or 16.

Because n is a multiple of 11, the difference between the two sums of its alternate digits is a multiple of 11. That is, $(3 + a + 6 + 0)-$ $(7 + b + 4) = a - b - 2$ is a multiple of 11. Because a and b are digits, this multiple is either 0 or -11 giving $b = a + 9$ or $b = a - 2$.

If $b = a + 9$, then $a = 0$ and $b = 9$. Then $a + b \neq 7$ and $a + b \neq 16$. Hence $b = a - 2$. Now, if $a + b = 7, a$ and $b$are not integers. Therefore $b = a - 2$ and $a + b = 16$. This gives $a = 9$ and $b = 7$.

Hence $n = 3797640$. It may be checked that n is divisible by all the integers from 1 to 12, inclusive.

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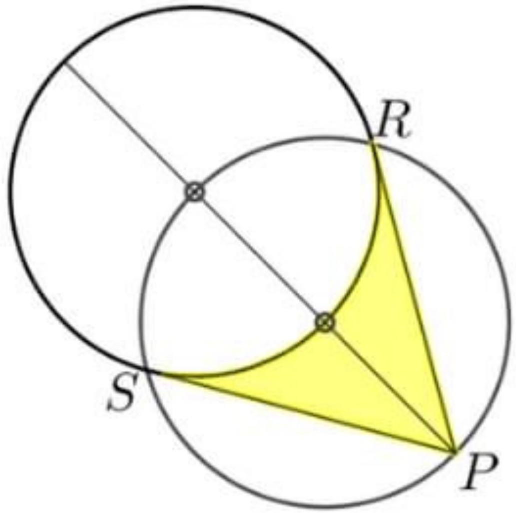
PROBLEM OF THE DAY 24



The diagram shows two circles of radius 1. Each circle passes through the centre of the other circle. The circles meet at the points R and S. The line through the centres of the circle meets one of the circles at the point P. PR and PS are tangents.

What is the area of the region coloured yellow?

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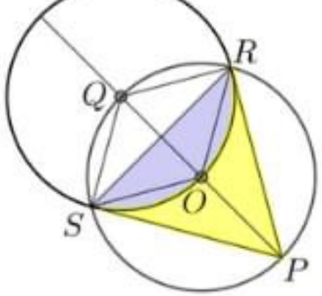
SOLUTION 24



$\sqrt{3} - \frac{1}{3}\pi$**.** We give an outline of the argument that shows this:

Let the points $O, Q, R,$ and S be as shown. The triangle ORQ is equilateral. Hence $\angle ROQ = 60°$. Similarly $\angle SOQ = 60°$. It follows that $\angle ROS = 120°$. Hence $\angle ROP = \angle POS = 120°$. Because they subtend equal angles at the centre, the chords PR, RS and SP are equal. Therefore the triangle PRS is equilateral.

From the triangle POR in which $PO = RO = 1$ and $\angle POR = 120°$ it follows that $RP = \sqrt{3}$. The area of an equilateral triangle with side length s is $\frac{\sqrt{3}}{4}s^2$. Therefore the area of PRS is $\frac{3\sqrt{3}}{4}$.

The area of the segment of the circle coloured blue is the area of the sector $QSOR$ less the area of the triangle SQR. Because $\angle SQR = 120°$ the area of the sector is $\frac{1}{3}\pi(1^2) = \frac{1}{3}\pi$. The triangle SQR is congruent to the triangle SOR and has area $\frac{\sqrt{3}}{4}$.

Therefore the area of the blue segment is $\frac{1}{3}\pi - \frac{\sqrt{3}}{4}$.

Finally, the area of the yellow region is the area of the triangle less the area of the blue segment, and hence is $\frac{3\sqrt{3}}{4} - \left(\frac{1}{3}\pi - \frac{\sqrt{3}}{4}\right) = \sqrt{3} - \frac{1}{3}\pi$.

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PROBLEM OF THE DAY 25

7 1 2 9 8 3 6 4 5

In the above sequence of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, there are increasing subsequences of length four, for example, 1 2 3 5, and also decreasing subsequences of length four, for example, 9 8 6 4.

Arrange the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 in a sequence, so that there is neither an increasing subsequence of length four, nor a decreasing subsequence of length four.

Can you do the same with the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10?

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SOLUTION 25

$$7\ 8\ 9\ 4\ 5\ 6\ 1\ 2\ 3$$

There is another solution with the numbers in the reverse of the above order.

It is not possible to arrange the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 without having either an increasing subsequence of length four, or a decreasing subsequence of length four.

This is a particular case of the Erdős-Szekeres Theorem. This says that each arrangement of n^2+1 numbers has either an increasing subsequence of length $n+1$ or a decreasing subsequence of length $n+1$.

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PROBLEM OF THE DAY 26

÷

Which is the smallest positive integer that is divisible by all of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17 and 18?

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Solution 26

12 252 240

The required number is the lowest common multiple of all the integers from 1 to 18. This is the number $2^4 \times 3^2 \times 5 \times 7 \times 11 \times 13 \times 17 = 12\,252\,240$.

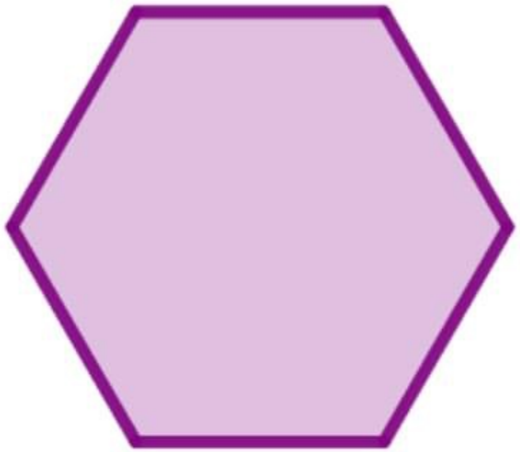
Curious fact:

What do you notice about the value of

$$12\,252\,240 \times 199\,?$$

And what do you notice if you multiply this last number by 2?

Problem Of The Day 27



In how many different ways can a regular hexagon be cut into two pieces of equal area by a single straight line cut?

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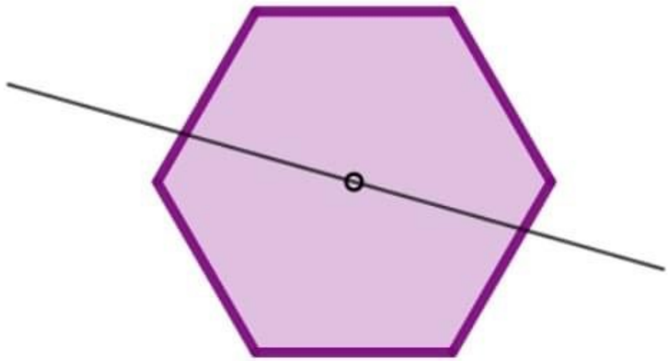
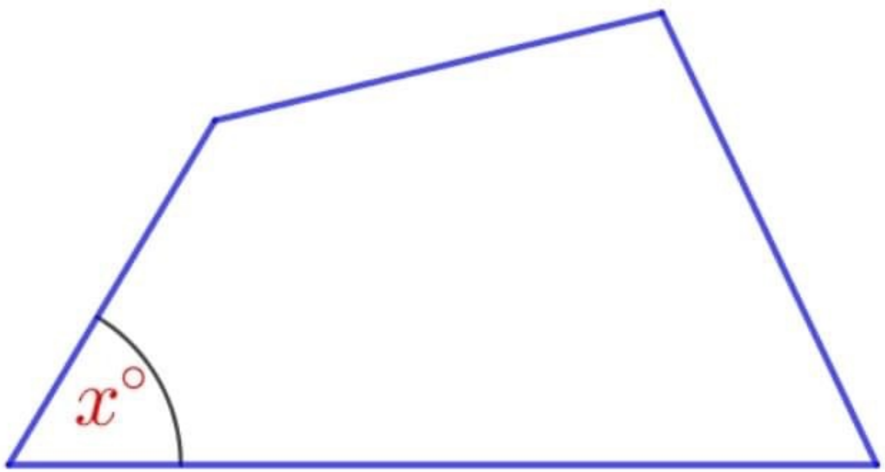
Solution 27

infinitely many



Each straight line through the centre of the hexagon divides it into two pieces of equal area. There are infinitely many of these lines.

Problem Of The Day 28



In this quadrilateral the angle x° is the mean of the other three angles.

What is the value of x?

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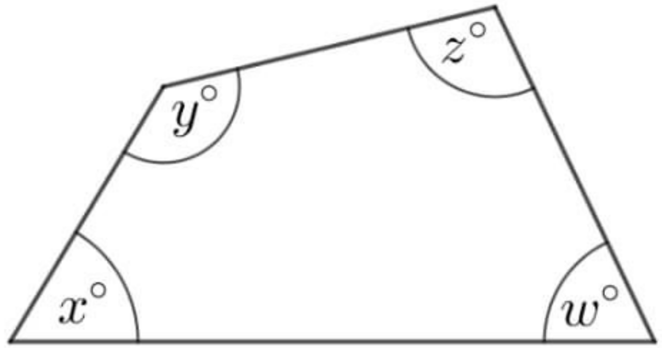
Solution 28

90



Let the other three angles of the quadrilateral be y°, z° and w°. The sum of the angles in a quadrilateral is 360°. Therefore $x + y + z + w = 360$. (1).
Because x is the mean of the other three angles, we have $x = \frac{1}{3}(y + z + w)$ and hence $y + z + w = 3x$.
Substituting for $y + z + w$ in (1) gives $x + 3x = 360$.
Hence $4x = 360$. Therefore $x = 90$.

Problem Of The Day 29

$$U \times K \times M \times T = 11111111$$

where U, K, M and T are four different prime numbers.

What is the value of $U + K + M + T$?

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Solution 29

322

$$\begin{aligned} 11111111 &= 1111 \times 10001 \\ &= 11 \times 101 \times 73 \times 137. \end{aligned}$$

Therefore U, K, M and T are $11, 73, 101, 137$ in some order. These numbers are all primes.

Therefore $U + K + M + T =$
$11 + 73 + 101 + 137 = 322.$

Problem Of The Day 30

$$\begin{array}{r} ONE \\ +\ ONE \\ +\ ONE \\ +\ ONE \\ \hline TEN \\ \hline \end{array}$$

In this strange looking sum, the letters E, N, O and T represent four different non-zero digits.

What is the value of $ONE + TEN$?

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Solution 30

910

Suppose that there is a carry of C from the tens column to the hundreds column. Then, from the units column [also known as the ones column] and the tens column we have that $4(10N + E) = 100C + 10E + N$. This rearranges to give

$$39N = 100C + 6N \quad (1).$$

Because $39N \leq 39 \times 9 = 351,$ it follows from (1) that $C \leq 4$. Also, by (1), $100C$ is a multiple of 3. Hence, $C = 0$ or $C = 3$. If $C = 0$ then, by (1) $39N = 6E$ and so $13N = 2E$. This equation has no solutions where N and E are digits.
Therefore $C = 3$. Hence from (1) it follows that $39N = 300 + 6E$. Therefore $13N = 100 + 2E$. Now $1 \leq E \leq 9$, and so $102 \leq 100 + 2E \leq 118$. The only multiple of 13 in this range is $104 = 8 \times 13$. Hence $E = 2,\ N = 8$ and $C = 3$.
Now, from the tens and hundreds columns, we have $T = 4O + 3$. Therefore, as $T \leq 9,$ we have $O = 1$ and $T = 7$.

Therefore $ONE + TEN = 182 + 728 = 910.$

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Problem Of The Day 31

I plan to make a model of the solar system on a scale that makes the diameter of the Sun 1 cm.

On this scale what, approximately, would be the distance of the nearest star Proxima Centauri from the Sun?

Note: The diameter of the Sun is approximately 1 400 000 km. Procima Centauri is 4.2 light years from the Sun. A light year is the distance that light travelling at 300 000 km per second travels in one year.

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Solution 31

284 km

There are about $3600 \times 24 \times 365$ seconds in a year. Therefore, in 4.2 years light travels $4.2 \times 3600 \times 24 \times 365 \times 300\,000$ km. Hence, the ratio of the distance of Proxima Centauri to the diameter of the Sun is

$$\frac{4.2 \times 3600 \times 24 \times 365 \times 300\,000}{1\,400\,000}.$$

This fraction cancels down to

$$0.3 \times 3600 \times 24 \times 365 \times 3 = 28\,382\,400.$$

Hence on a scale on which the Sun's diameter is 1 cm Proxima Centauri would be 28 382 400 cm from the Sun, that is, about 284 km.

Problem Of The Day 32

Tom, Dick and Harry work packing toys.
When just Tom and Dick are working it takes them 36 minutes to fill the required number of boxes. When just Dick and Harry are working it takes them 45 minutes to fill the same number of boxes. When just Harry and Tom are working they take 1 hour to fill the same number of boxes.

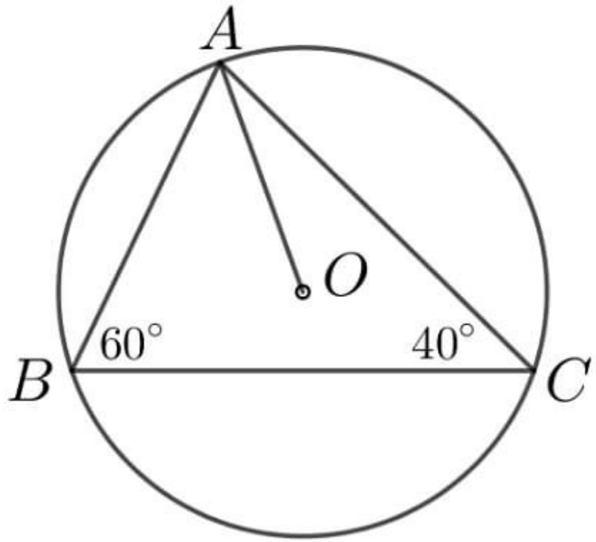
How long does it take them to fill the same number of boxes when Tom, Dick and Harry are all working?

Solution 32

half an hour (30minutes)

When just Tom and Dick are working it takes them 36 minutes to fill the required number of boxes. Therefore in one hour they would fill $\frac{60}{36} = \frac{5}{3}$ times the required quota. Similarly in one hour Dick and Harry both working would fill $\frac{60}{45} = \frac{4}{3}$ times the quota, and Tom and Harry would fill $\frac{60}{60} = 1$ times the quota. Therefore if all three work for an hour they would fill $\frac{1}{2}(\frac{5}{3} + \frac{4}{3} + 1) = 2$ times the quota. Hence it would take them half an hour to fill the quota.

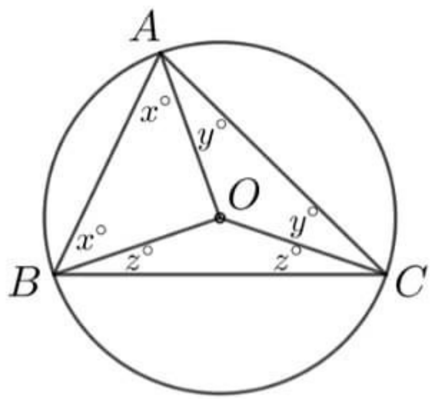
Problem Of The Day 33



The vertices of the triangle ABC are on the circle with centre O.
$\angle ABC = 60^\circ$ and $\angle ACB = 40^\circ$.

What is $\angle BAO$?

Solution 33



50°

Let $\angle BAO = x^\circ$. Because O is the centre of the circle, $OA = OB$. Therefore the triangle AOB is isosceles. Hence $\angle ABO = \angle BAO = x^\circ$. Similarly $\angle CAO = \angle ACO$ $= y^\circ$, say, and $\angle CBO = \angle BCO = z^\circ$, say.

The angles of the triangle ABC have sum 180°. Therefore $2x^\circ + 2y^\circ + 2z^\circ = 180^\circ$, and hence $x^\circ + y^\circ + z^\circ = 90^\circ$. We also have $y^\circ + z^\circ = \angle ACO + \angle BCO = \angle ACB = 40^\circ$. Hence $x^\circ = (x^\circ + y^\circ + z^\circ) - (y^\circ + z^\circ) = 90^\circ - 40^\circ = 50^\circ$.
[A good alternative method is to use the facts that $\angle AOB = 2 \times \angle ACB = 80^\circ$, and that the triangle AOB is isosceles. It follows that $x^\circ = \frac{1}{2}(180 - 80)^\circ = 50^\circ$.]

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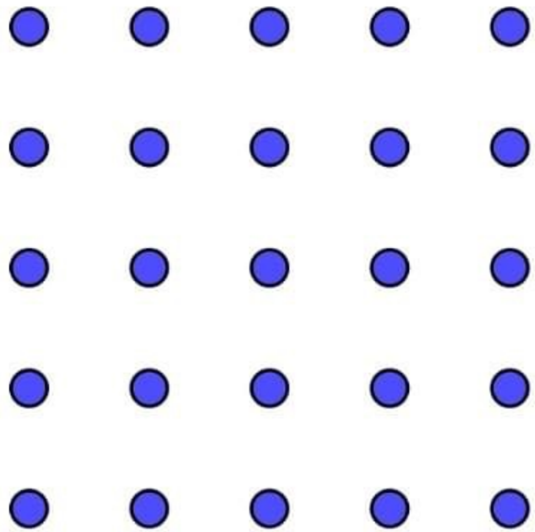
Problem Of The Day 34



How many different straight lines are there that go through exactly three points of this 5×5 array of points?

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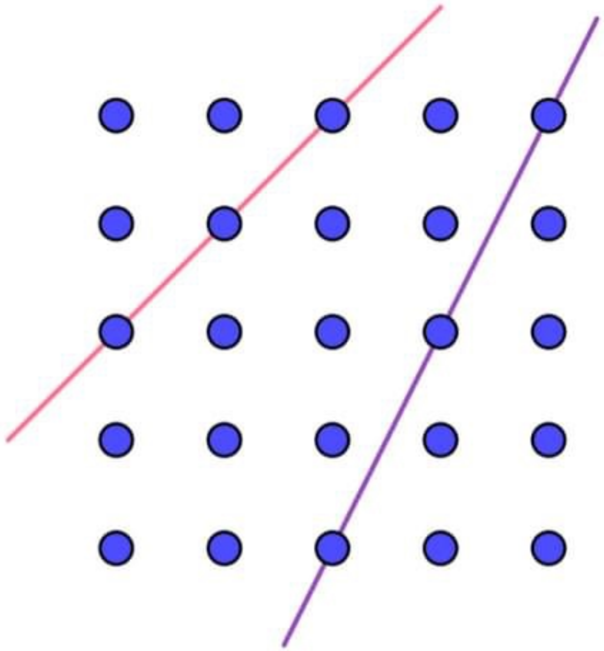
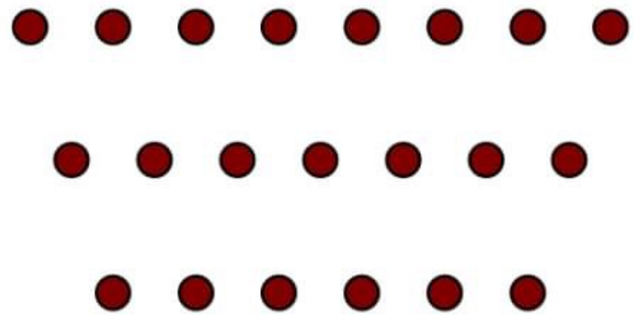
Solution 34



16

There are 4 lines similar to the one shown in red, and 12 lines similar to the one shown in purple. This makes a total of 16 lines.

Problem Of The Day 35



The game NIM is played with rows of coins. There are two players who move alternately. A move consists of removing one or more coins that all come from the same row. The player who takes the last coin wins.

In the above position it is your turn to move. Which is the only move to ensure that you win?

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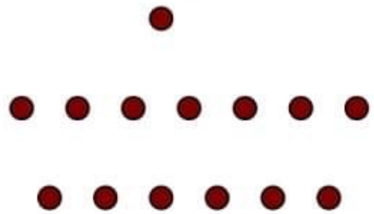
Solution 35

Take 7 coins from the top row.

This leaves the following position:



You can now force a win whatever your opponent does.
The idea behind this is as follows.

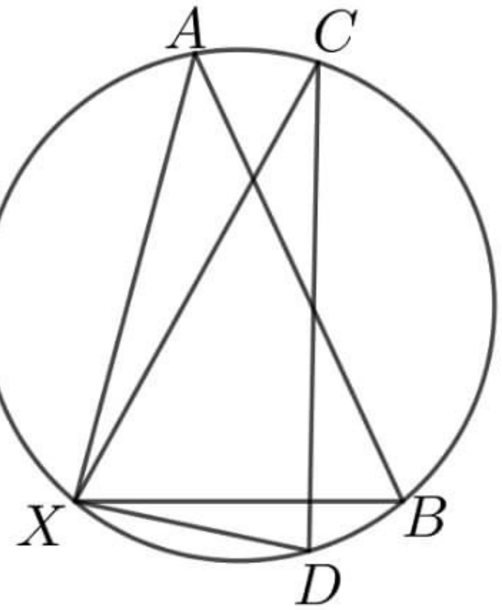
We say that a position is *good* if, when you write the number of coins in each row in binary, one under another, the number of 1s in each column is even. You can force a win if you can move to create a good position.

You can see that the above position is good:

$$1 = 0\,0\,1$$
$$7 = 1\,1\,1$$
$$6 = 1\,1\,0$$

However your opponent moves, the new position will not be good. You can then move to make it good again, and so on. The position after the last coin has been taken is good. If you follow the strategy outlined above, this position will will occur after your move. So you win.

Problem Of The Day 36

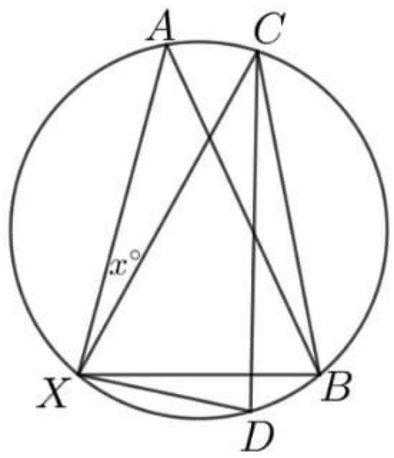


A, B, C, D and X are points on a circle.
$\angle BXA = \angle DXC = 75°$, $\angle BAX = 40°$
and $\angle DCX = 35°$.

What is $\angle CXA$?

Solution 36

5°



Let $\angle CXA = x^\circ$.
The sum of the angles in a triangle is 180°. Therefore, from the triangle AXB, we have
$\angle ABX = 180^\circ - \angle BAX - \angle BXA = 180^\circ - 40^\circ - 75^\circ = 65^\circ$.
Similarly, from the triangle CXD,
$\angle CDX = 180^\circ - \angle DCX - \angle DXC = 180^\circ - 35^\circ - 75^\circ = 70^\circ$.
Angles subtended by the same chord at the circumference of a circle are equal. Therefore $\angle CBA = \angle CXA = x^\circ$ and
$\angle CBX = \angle CDX = 70^\circ$.
Hence $x^\circ = \angle CBA = \angle CBX - \angle ABX = 70^\circ - 65^\circ = 5^\circ$.

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Problem Of The Day 37

“How much are the apples?”, Granny Smith asked.
“The first apple costs 52p, the second 51p, and so on”, replied the greengrocer. “Each apple will cost you a penny less than the previous apple.”

Granny Smith spent £10 on apples.
How many did she buy?

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Solution 37

25

£10 = 1000p. So one way to find the answer is to do the sum $52 + 51 + 50 + \ldots$ until you reach a total of 1000. However, it saves work to use some algebra.

Suppose $x + 1$ is the cost, in pence, of the cheapest apple. Then we require that $(x+1) + (x+2) + \ldots + 52 = 1000$. This is equivalent to $(1 + 2 + \ldots + 52) - (1 + 2 + \ldots + x) = 1000$. Using the formula $1 + 2 + \ldots + k = \frac{1}{2}k(k+1)$ for the sum of the first k positive integers, this is equivalent to $\frac{1}{2}(52 \times 53) - \frac{1}{2}x(x+1) = 1000$. That is, $1378 - \frac{1}{2}x(x+1) = 1000$. This last equation may be rearranged to give $x^2 + x - 756 = 0$, which is equivalent to $(x-27)(x+28) = 0$, with the positive solution $x = 27$.

It follows that Granny Smith bought $52 - 27 = 25$ apples.

Problem Of The Day 38

Five years ago, Pat was a quarter of her mother's age.
When Pat's mother is four times the age Pat is now,
Pat's mother will be twice Pat's age.

How old is Pat now?

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Solution 38

15

Let the ages of Pat and her mother now be p and m years, respectively.

Five years ago, Pat was a quarter of her mother's age. Therefore $p - 5 = \frac{1}{4}(m - 5)$. This equation may be rearranged to give $m = 4p - 15$. (1)

Pat's mother will be four times Pat's age now when her age is $4p$. This will be in $4p - m$ year's time. Pat's age then will be $p + (4p - m) = 5p - m$. Because Pat will then be a half of her mother's age, $5p - m = \frac{1}{2}(4p)$. Therefore $m = 3p$. Hence, substituting for m in (1), we deduce that $3p = 4p - 15$. Therefore $p = 15$.

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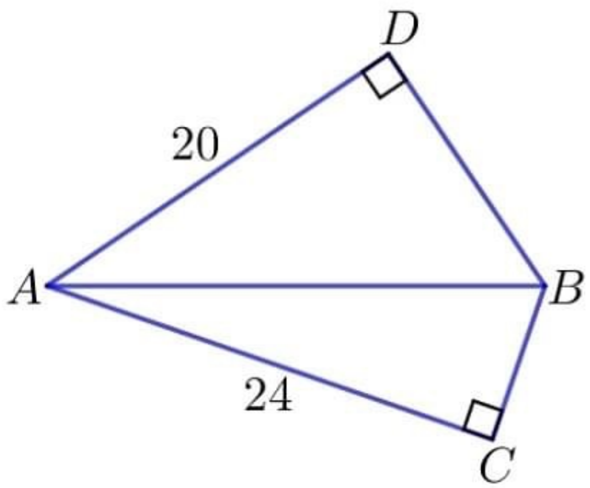
Problem Of The Day 40

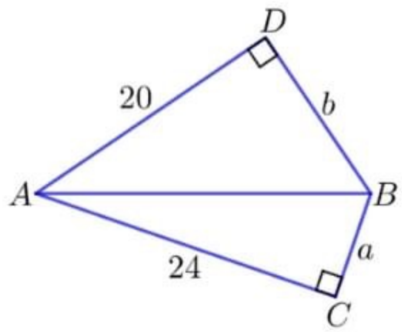


ABC and ABD are non-congruent right -angled triangles with integer side lengths. AC has length 24, and AD has length 20.

What is the length of AB?

Solution 40

25



Let BC have length a and BD have length b.
By Pythagoras' Theorem $AB^2 = 24^2 + a^2 = 20^2 + b^2$.
It follows that $b^2 - a^2 = 24^2 - 20^2 = 176$. Hence $(b - a)(b + a) = 176$. Now $b - a$ and $b + a$ are positive integers with $b - a < b + a$. Also $b - a$ and $b + a$ differ by $2a$ and hence are both even or both odd. So the only possibilities for the pair $(b - a, b + a)$ are $(2, 88)$, $(4, 44)$ and $(8, 22)$. Hence, we have either $b = 45, a = 43$ or $b = 24, a = 20$, or $b = 15, a = 7$. Now, If $b = 45$, $AB^2 = 20^2 + 45^2 = 2425$ and so $AB = 5\sqrt{97}$ which is not an integer. If $b = 24$ and $a = 20$, the triangles ABC and ABD are congruent. Hence $b = 15$ and $a = 7$. Hence $AB^2 = 20^2 + 15^2 = 625$. Therefore $AB = 25$.

Problem Of The Day 41

$$a + b + c$$

a, b and c are three different digits. They may be arranged to make six three-digit numbers. The sum of these six numbers is 3552.

What is the value of $a + b + c$?

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Solution 41

16

If x, y and z are digits, the expression "xyz" stands for the number $100x + 10y + z$. Therefore the sum of the six three-digit numbers that may be made using the the digits a, b and c is

$(100a + 10b + c) + (100a + 10c + b) + (100b + 10a + c)$

$+(100b + 10c + a) + (100c + 10a + b) + (100c + 10b + a)$

$= 222(a + b + c)$. Therefore $222(a + b + c) = 3522$.

Hence $a + b + c = \dfrac{3552}{222} = 16$.

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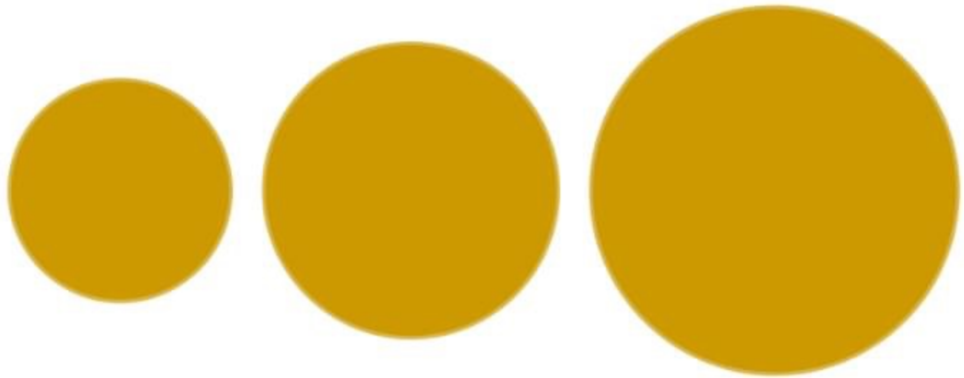
Problem Of The Day 42



Mrs Pastry made three circular buns for her four children. The buns had the same height, but their radii were 3 cm, 4 cm and 5 cm.

Show how she could cut the buns into a total of not more than five pieces which she could then share between her children, so they all received the same amount.

[This puzzle was devised by Henry Dudeney (1857-1930), a great creator of puzzles.]

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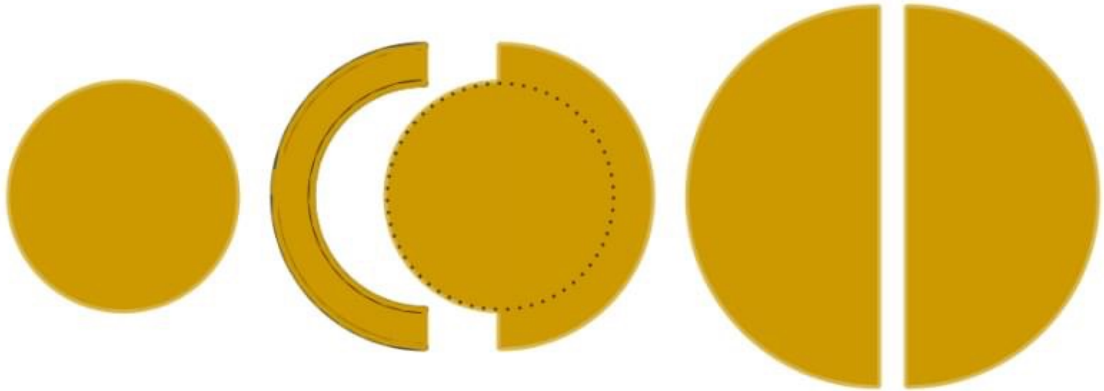
Solution 42



Because $\pi 3^2 + \pi 4^2 = \pi 5^2$, the total volume of the two smaller buns equals the volume of the largest bun. Therefore each child's share should be equal to half of the largest bun. One way to achieve this is shown above. The dotted circle has radius 2 cm. One child should be given the two pieces on the left. The other children should each be given one of the other pieces.

Other solutions are possible.

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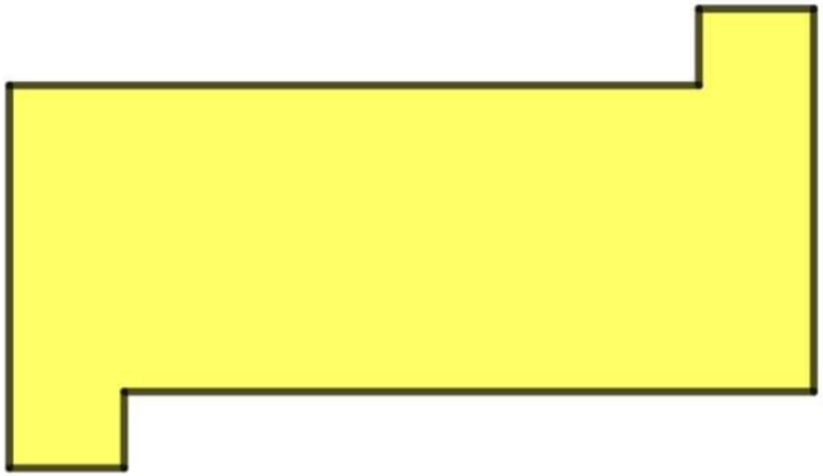
Problem Of The Day 44



Show how the above octagon may be cut into two pieces that can be rearranged to make a rectangle.
[The shape is made up of an $8\,\text{cm} \times 21\,\text{cm}$ rectangle, together with two $2\,\text{cm} \times 3\,\text{cm}$ rectangles.]

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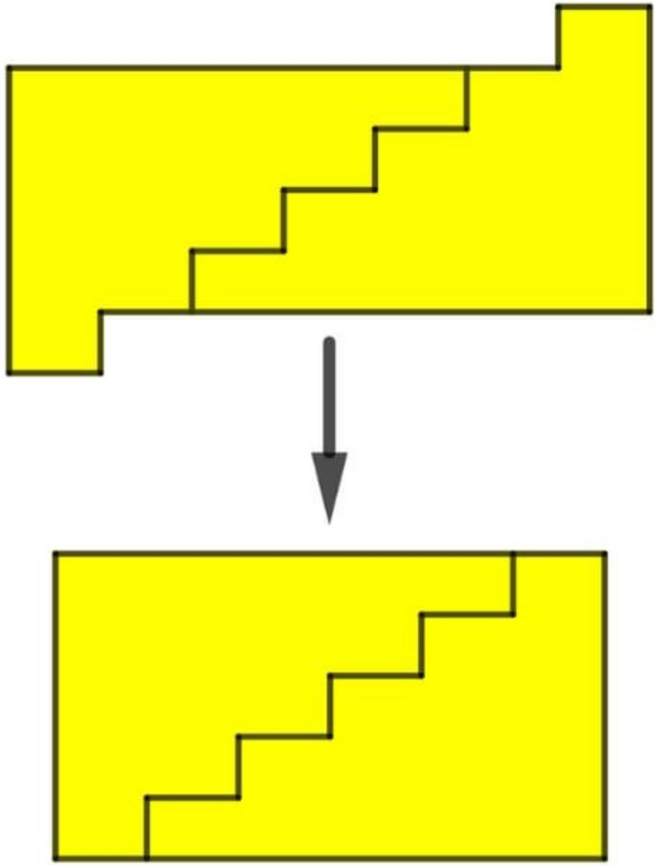
Solution 44



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Problem Of The Day 45

$a+b+c?$

a, b and c are three different positive integers with the following properties:

(i) none of these integers is a square,

(ii) the product of each pair of these integers is a square,

and

(iii) $a+b+c$ has the least value among all triples of positive integers that satisfy (i) and (ii).

What is the value of $a+b+c$?

Solution 45

28

It can be shown that a triple of positive integers that satisfies (i) and (ii) has the form qr^2, qs^2, qt^2, where q is an integer, with $q > 1$, that is not divisible by any square other than 1, and r, s and t are three different positive integers.

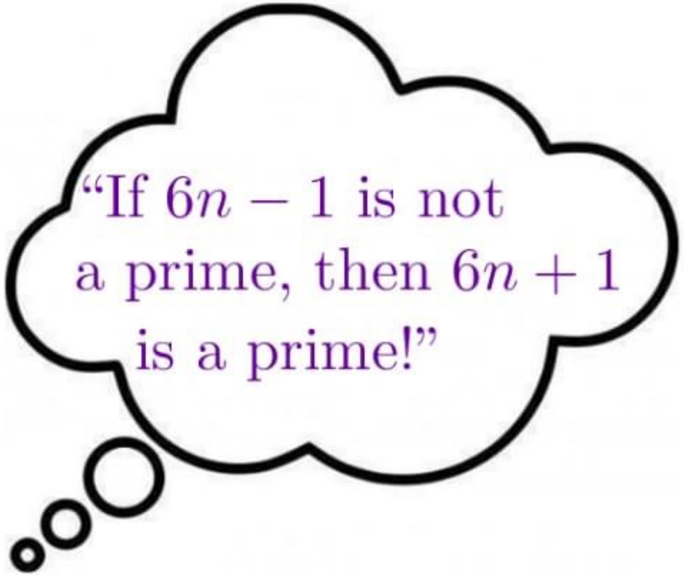
The sum of these three integers is $q(r^2 + s^2 + t^2)$. The least possible value of q is 2, and the least possible values of r, s and t are 1, 2 and 3 in some order.

Therefore the least possible value of $a + b + c$ is

$2(1^2 + 2^2 + 3^2) = 2(1 + 4 + 9) = 28.$

[This gives a, b and c the values 2, 8 and 18. Hence ab, ac and bc are the squares 16, 36 and 144.]

Problem Of The Day 46



Alan thought that for each positive integer n, if the number $6n-1$ is not a prime, then the number $6n+1$ will be a prime.

Give a counterexample to show that this is *NOT* true.

Solution 46

20 or 24 or 31 or . . .

A counterexample to the statement is a positive integer n such that $6n - 1$ is not prime and yet $6n + 1$ is also not prime. The smallest counterexample is $n = 20$.

$119 = 6 \times 20 - 1$ and is not prime $(119 = 7 \times 17)$, but also $121 = 6 \times 20 + 1$ is not prime $(121 = 11^2)$.

$n = 24$ and $n = 31$ also provide counterexamples. In fact, there are infinitely many counter examples. That is, there are infinitely many positive integers n such that both $6n - 1$ and $6n + 1$ are not prime. Can you prove this?

Note: It is known, but not easy to prove, that there are infinitely many primes of the form $6n - 1$ and of the form $6n + 1$. It is not known whether there are infinitely many positive integers n for which both $6n - 1$ and $6n + 1$ are prime.

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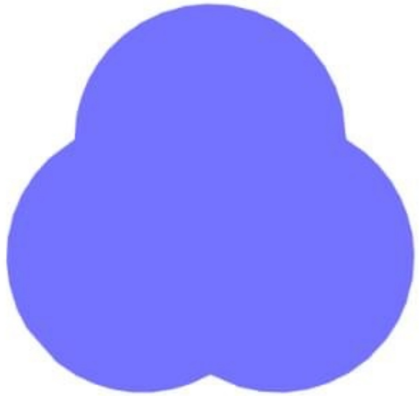
Problem Of The Day 47



Three circles each have radius 1 cm and pass through the centres of the other two circles?

What is the area of the shape that they enclose?

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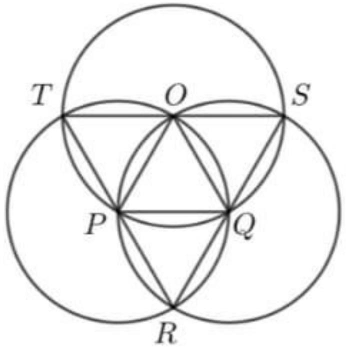
Solution 47

$(\sqrt{3} + \frac{3}{2}\pi)$ cm^2



Let O, P and Q be the centres of the circles and let R, S and T be the other points where the circles meet. The triangles OPQ, OQS, OTP and PRQ are all equilateral. It follows that $\angle SOT = 3 \times 60^\circ = 180^\circ$. Therefore SOT is a straight line and hence a diameter, and likewise for RQS and TPR.

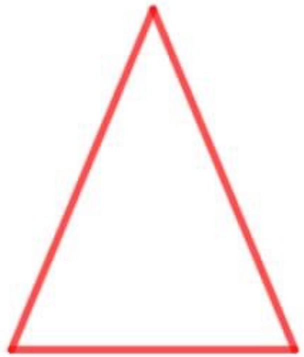
Therefore, the shape is made up of the equilateral triangle RST with side length 2 cm, and the three semi-circles with diameters RS, ST and TR, each with radius 1 cm, that lie outside the triangle.

The triangle RST has height $\sqrt{3}$ cm and a base of length 2 cm, and hence an area of $\frac{1}{2}(2 \times \sqrt{3})$ cm^2, that is, $\sqrt{3}$ cm^2.

Each of the three semicircles has area $\frac{1}{2}\pi$ cm^2.

Therefore the total area is $(\sqrt{3}+\frac{3}{2}\pi)$ cm^2.

Problem Of The Day 48

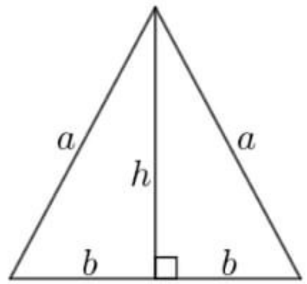


The area of a particular isosceles triangle, in cm^2, has the same numerical value as the length, in cm, of its perimeter.
The length of the perimeter, in cm, and the length of the base, in cm, are both integers.

What is the area of the triangle?

Solution 48

27 cm²



Let the lengths of the sides of the triangle and its height, in cm, be as shown. Then the length of the perimeter, in cm, is $2(a+b)$.
By Pythagoras' theorem $h = \sqrt{a^2 - b^2}$. It follows that the area of the triangle, in cm^2, is $b\sqrt{a^2 - b^2}$.
These are numerically equal if $2(a+b) = b\sqrt{a^2 - b^2}$.
By squaring this last equation we obtain
$4(a+b)^2 = b^2(a^2 - b^2)$, that is, $4(a+b)^2 = b^2(a-b)(a+b)$.
Since $a + b \neq 0$, this is equivalent to $4(a+b) = b^2(a-b)$.
This last equation may be rearranged to give $a = \dfrac{b(b^2+4)}{b^2-4}$.
Hence the common numerical value of the area, in cm^2, and the perimeter, in cm, is given by $A = \dfrac{4b^3}{b^2-4}$. It can be checked that the only value of b for which both $2b$ and A are positive integers is 6 which makes $A = 27$.

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Problem Of The Day 49

$$a! \times b! = c!$$

Find a solution of this equation, where a, b and c are positive integers, with $a < b < c$.

Show that there are infinitely many solutions of this equation.

Here $n!$, pronounced "n factorial", is the product of all the integers from 1 to n inclusive.
For example, $7! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 = 5040$.

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Solution 49

one solution is $a = 3, \ \ b = 5, \ \ c = 6$.

We have

$$
\begin{aligned}
3! \times 5! &= (1 \times 2 \times 3) \times (1 \times 2 \times 3 \times 4 \times 5) \\
&= 6 \times (1 \times 2 \times 3 \times 4 \times 5) \\
&= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \\
&= 6!.
\end{aligned}
$$

This generalizes to give $n! \times (n! - 1)! = (n!)!$ for each positive integer n, with $n > 2$. Therefore, we have the infinitely many solutions $a = n, b = n! - 1, c = n!$, for $n = 3, 4, 5, \ldots$ There is also the solution $a = 6, b = 7, c = 10$ that is not of the above type. It is not known if there are any other solutions. However, it has been proved that there are no other solutions with $b \leq 10^{3000}$.

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Problem Of The Day 50

$$300^{300} - 1$$

Which is the smallest prime that is a factor of $300^{300} - 1$?

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Solution 50

7

Because 2, 3 and 5 are factors of 300, they are also factors of 300^{300}. Therefore the primes 2, 3 and 5 are not factors of $300^{300} - 1$.

We next consider 7 as a possible factor. We first note that $301 = 7 \times 43$ and so 7 is a factor of 301.
Now, $300^2 - 1 = 300^2 - 1^2 = (300 - 1)(300 + 1)$
$= 299 \times 301$. Therefore 301 is a factor of $300^2 - 1$ and hence 7 is a factor of $300^2 - 1$.

For each positive integer n, $x = 1$ is a solution of the equation $x^n - 1 = 0$. Therefore $x - 1$ is a factor of $x^n - 1$.
If we now put $x = 300^2$ and $n = 150$, we deduce that $300^2 - 1$ is factor of $(300^2)^{150} - 1$, that is, of $300^{300} - 1$.
Because 7 is a factor of $300^2 - 1$, we deduce that 7 is a factor of $300^{300} - 1$. Therefore 7 is the smallest prime that is a factor of $300^{300} - 1$.

Problem Of The Day 51

$$1^2 + 3^2 + \ldots + 97^2 + 99^2$$

What is the sum of the squares of the first fifty odd positive integers?

Note : The formula for the sum of the squares of the first n positive integers is

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

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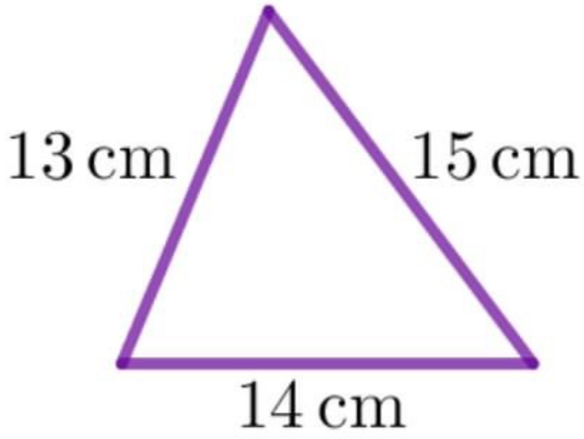
Solution 51

166 650

We have, using the formula given in the question,

$$\begin{aligned}
1^2 + 3^2 + \ldots + 97^2 + 99^2 = \\
&(1^2 + 2^2 + 3^2 + 4^2 + \ldots + 97^2 + 98^2 + 99^2 + 100^2) \\
&-(2^2 + 4^2 + \ldots + 98^2 + 100^2) \\
&= (1^2 + 2^2 + 3^2 + 4^2 + \ldots + 97^2 + 98^2 + 99^2 + 100^2) \\
&\quad -4(1^2 + 2^2 + \ldots + 49^2 + 50^2) \\
&= \frac{100 \times 101 \times 201}{6} - 4 \times \frac{50 \times 51 \times 101}{6} \\
&= \frac{101}{6}(100 \times 201 - 200 \times 51) = \frac{101 \times 100}{6}(201 - 2 \times 51) \\
&= \frac{101 \times 100 \times 99}{6} = 101 \times 50 \times 33 = 166\,650.
\end{aligned}$$

Problem Of The Day 52



The triangle shown has sides with lengths 13 cm, 14 cm and 15 cm.

What, in cm^2, is the area of this triangle?

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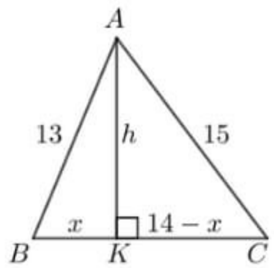
Solution 52

84 cm^2



Let A, B, C and K, be as shown in the diagram. Let h be the height, in cm, of the triangle and let $BK = x\,\text{cm}$. Then $KC = (14 - x)\,\text{cm}$.

By Pythagoras' Theorem applied to the triangles ABK and ACK, $h^2 + x^2 = 13^2$ and $h^2 + (14 - x)^2 = 15^2$. Subtracting the first equation from the second gives $(14 - x)^2 - x^2 = 15^2 - 13^2$. This simplifies to give $28x = 140$, and so $x = 5$. Therefore $h^2 = 13^2 - 5^2 = 144$, and so $h = 12$. Hence the area of the triangle, in cm^2, is $\frac{1}{2}(14 \times 12) = 84$.

Note: This method applied to a general triangle with side lengths a, b and c, gives Heron's formula, $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a + b + c)$, for the area of a triangle.
This question may also be answered using trigonometry.

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Problem Of The Day 53

What is the largest sum of money you could have in coins, and still not be able to give exact change for a £10 note?

Note: By "coins" we mean current UK coins in common circulation. So you should ignore the gold Sovereign and the £5 Crown.

Solution 53

£10.43

There are several ways in which you could have £10.43 in coins and still not be able to give exact change for a £10 note. One of them is:

four £2 coins, one £1 coin, one 50p coin,
four 20p coins, one 5p coin and four 2p coins.

Problem Of The Day 54

!

Find the highest power of 3 that is a factor of

$$1! + 2! + 3! + \ldots + 98! + 99! + 100!$$

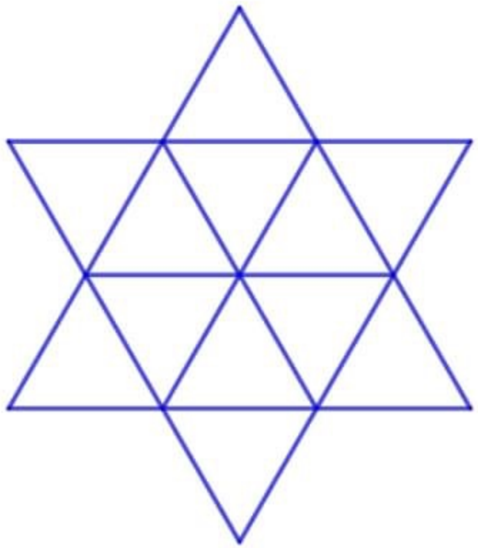
Here $n!$, pronounced "n factorial", is the product of all the integers from 1 to n inclusive.
For example, $7! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 = 5040$.

Solution 54

$9 = 3^2$

For convenience we put $S = 1! + 2! + 3! + \ldots + 98! + 99! + 100!$.
Because $9! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9$, 27 is a factor of 9!
Similarly 27 is a factor of $n!$ for all integers $n \geq 9$. Also, we have
$7! + 8! = 6!(7 + 7 \times 8) = 720 \times 63 = 27 \times 1680$. Therefore
$7! + 8! + 9! + \ldots + 98! + 99! + 100! = 27N$, where N is a (very large) integer.
Now, $1! + 2! + 3! + 4! + 5! + 6! = 1 + 2 + 6 + 24 + 120 + 720$
$= 873 = 9 \times 97$. Hence $S = 9 \times 97 + 27N = 9(97 + 3N)$.
It follows that 9 is a factor of S. However $27 (= 3^3)$ is not a factor of S. For suppose 27 were a factor of S. Then, as $S = 9(97 + 3N)$, it would follow that 3 is a factor of $97 + 3N$, and hence that 3 is a factor of 97. Since 3 is not a factor of 97, we deduce that 27 is not a factor of S.
Hence 9 is the highest power of 3 that is a factor of S.

Problem Of The Day 55



Put the integers 1 to 12 in the triangles, with one integer in each triangle, so that each sum of the integers in two triangles which share an edge is a prime.

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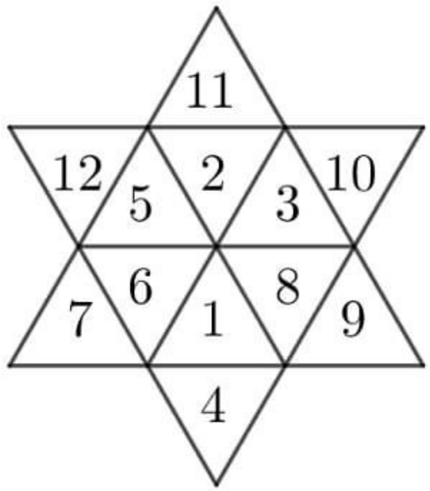
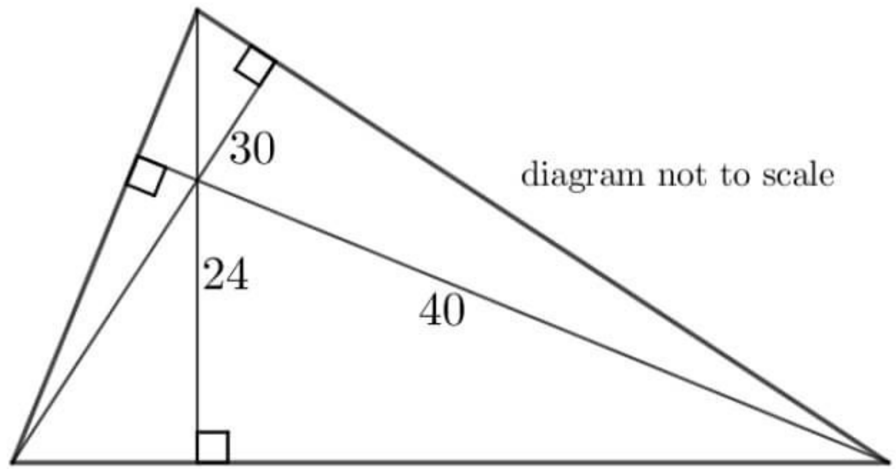
Solution 55



There are very many other solutions.

If you found this puzzle easy, try to solve the same problem, first using the twelve integers from 17 to 28, inclusive, and then with the integers 18 to 29. Again, in each case the sum of the two numbers in adjacent triangles should always be prime.

Problem Of The Day 56

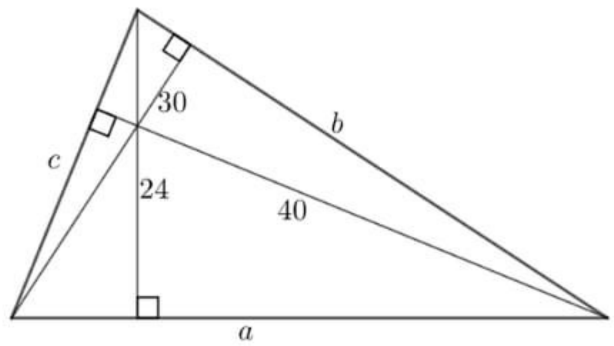


The altitudes of the triangle shown have lengths, in cm, 24, 30 and 40.

What is the area, in cm^2, of the triangle?

Solution 56

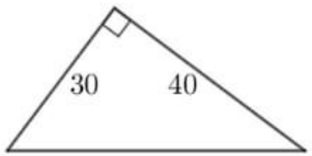
600 cm²



Let the side lengths of the triangle, in cm, be a, b and c, as shown. Let the area of the triangle, in cm^2, be A. Because the area of a triangle is $\frac{1}{2}(\text{base} \times \text{height})$, we have $A = \frac{1}{2}(24a) = \frac{1}{2}(30b) = \frac{1}{2}(40c)$. Therefore

$$a : b : c = \frac{A}{12} : \frac{A}{15} : \frac{A}{20} = \frac{1}{12} : \frac{1}{15} : \frac{1}{20} = 5 : 4 : 3.$$

Since $5^2 = 3^2 + 4^2$, it follows, by the converse of Pythagoras' Theorem, that the triangle is right-angled. Hence its two shortest sides are the altitudes with lengths, in cm, of 30 and 40.



Therefore the area of the triangle, in cm^2, is $\frac{1}{2}(30 \times 40) = 600$.

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Problem Of The Day 57

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89 \ldots$$

Find a set of Fibonacci numbers that between them contain each of the ten digits from 0 to 9 once and once only.

Note: The Fibonacci numbers are the numbers in the infinite sequence which starts $1, 1,$ and in which every other term is the sum of the previous two terms.

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Solution 57

2, 5, 34, 610, 987

These are the 3rd, 5th, 9th, 15th and 16th Fibonacci numbers. This is the only solution.

Can you find the largest Fibonacci number with no repeated digits?

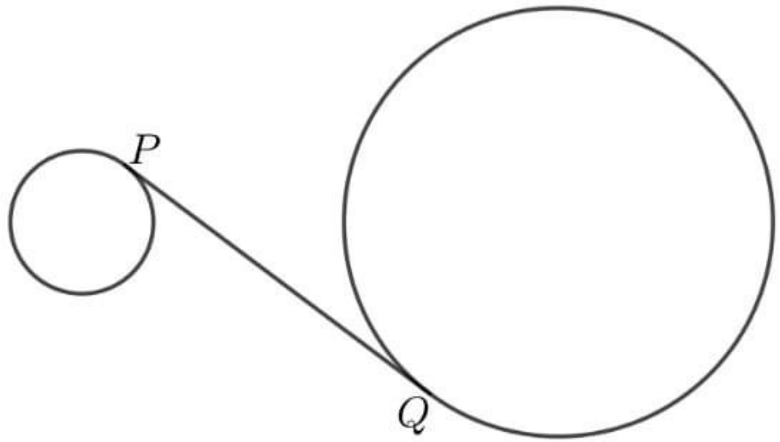
Myths about Fibonacci:

1. "He was called Fibonacci." No, his name was Leonardo of Pisa. He lived from 1170 to 1250. It seems that the nickname Fibonacci was invented by Guillaume Libri in 1838.

2. "He discovered the Fibonacci numbers." No, it is thought that they first appeared in the *Chandahshastra*, a Sanscrit book written about 2300 years ago.

See *The Man of Numbers*, Keith Devlin, Bloomsbury, 2011, for more details.

Problem Of The Day 58



The diagram shows two circles with radii 3 units and 9 units, and whose centres are 20 units apart. The line segment PQ is a tangent to the circles at P and Q.

What is the length of PQ?

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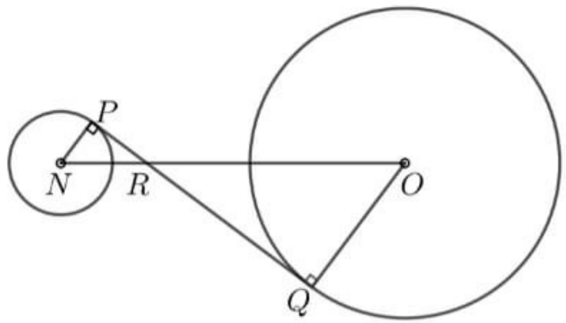
Solution 58

16



Let the centres of the circles be N and O as shown, and let R be the point where NO meets PQ.
A tangent to a circle is at right angles to the radius at the point of tangency. Therefore $\angle NPR = \angle OQR = 90^\circ$. The vertically opposite angles $\angle NRP$ and $\angle ORQ$ are equal. Therefore the triangles NRP and ORQ are similar. Hence $\dfrac{NR}{OR} = \dfrac{NP}{OQ} = \dfrac{3}{9} = \dfrac{1}{3}$.
Therefore $NR = \frac{1}{4}NO = \frac{1}{4}(20) = 5$ and so $OR = 15$.
Hence, by Pythagoras' Theorem applied to the triangles NRP and ORG, we have $PR = \sqrt{5^2 - 3^2} = 4$ and $QR = \sqrt{15^2 - 9^2} = 12$.
Therefore $PQ = PR + QR = 4 + 12 = 16$.

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Problem Of The Day 59

Postie Pat has ten letters for ten different addresses. On a bad day Pat delivers them entirely at random, except that one letter is delivered to each of the ten addresses.

What is the probability that exactly two letters are delivered to the wrong address?

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Solution 59

1 in 80 640

Suppose that the letter addressed to house A is delivered to house B. Then both A and B receive the wrong letter. Therefore, if exactly two letters are wrongly delivered, the letter for house B must be delivered to house A. Thus the only way exactly two letters can be delivered to wrong addresses is for a pair of letters to be swapped, with all the other letters going to the correct addresses. The number of ways in which this can happen is the the number of ways of choosing a pair of letters from the eight letters. This number is "10-choose-2" = $\binom{10}{2} = \frac{10 \times 9}{2 \times 1} = 45.$

The total number of ways in which ten letters can be delivered to ten addresses is $10! = 3\,628\,800$. Therefore the probabilty that exactly two letters go to the wrong address is $\frac{45}{3\,628\,800} = \frac{1}{80\,640}$.

Problem Of The Day 60

$$\begin{array}{r} THREE \\ +\ THREE \\ +\ THREE \\ \hline SEVEN \\ \hline \end{array}$$

In this sum which, despite appearances, is correct, the letters E, H, N, R, S, T and V represent seven different digits.

Find the numbers represented by $THREE$ and $SEVEN$.

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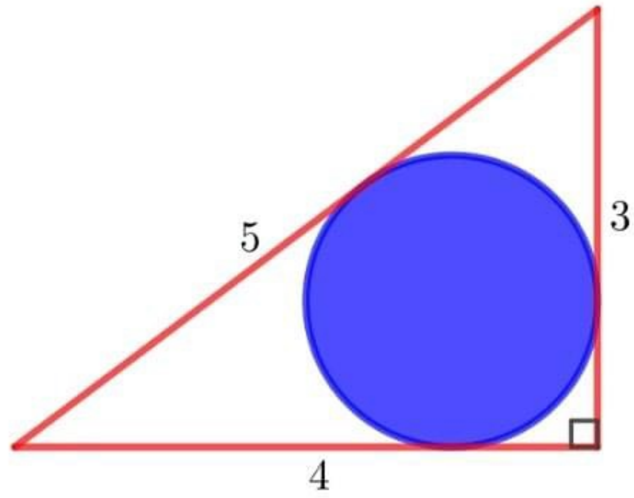
Solution 60

23199 and 69597

It may be checked that the sum is:

$$\begin{array}{r} 23199 \\ +\ 23199 \\ +\ 23199 \\ \hline 69597 \\ \hline \end{array}$$

Problem Of The Day 61

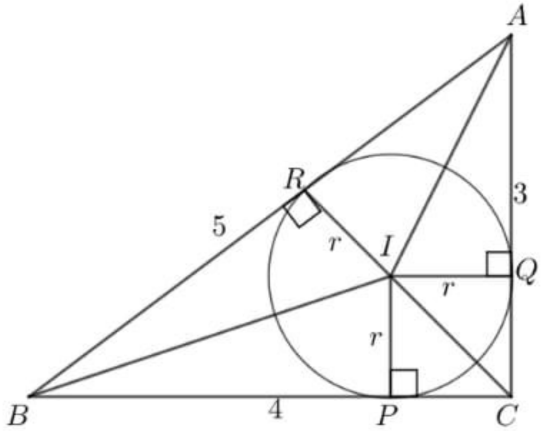


A circle touches the three sides of a right-angled triangle that has sides of lengths 3, 4 and 5.

What is the radius of the circle?

Solution 61

1



Let A, B and C be the vertices of the triangle, I be the centre of the circle, and P, Q and R be the points where the circle touches the sides of the triangle. Let r be the radius of the circle.

Because a tangent to a circle is perpendicular to the radius at the point of tangency, the angles shown are right angles. Therefore, using the formula area $= \frac{1}{2}$(height $\times$ base), we see that the triangles AIB, BIC and CIA have areas $\frac{5}{2}r$, $2r$ and $\frac{3}{2}r$, respectively. The total of the areas of these triangles equals the area of the triangle ABC. Therefore, we have $\frac{5}{2}r + 2r + \frac{3}{2}r = \frac{1}{2}(3 \times 4)$. Hence $6r = 6$. Therefore $r = 1$.

Problem Of The Day 62

1 2 3 4 5 6 7 8 9 10

In how many ways is it possible to choose three integers from 1 to 10, inclusive, with no two being consecutive?

1) We have 10C3 = 120 different ways to select 3 numbers.

2) We have 8 different ways such that the three numbers are consecutives.

3) Only two consecutive numbers are selected

If the pairs (1, 2) & (9, 10), , we have seven ways to select the 3rd number.

For the pairs (3,4), (4,5), ..., (8, 9) s we have 6 ways to select the 3rd number

→ The answer is 120 – 2*7 – 6*6 = 70 ways.

Problem Of The Day 63

$$\sqrt{x}$$

Which positive number is increased by 200% when you take its square root?

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Problem Of The Day 64

117 people enter a singles knock-out tennis tournament. In each round, if there is an odd number of players who have not yet been knocked out, one of them has a bye and automatically progresses to the next round. The others play each other in pairs, and the winner goes forward.

How many matches have been played by the time the final winner emerges?

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Solution 64

116

Each match results in one player being knocked out. For a single winner to emerge from the 117 players, 116 need to be knocked out. So 116 matches need to be played to produce the final winner.

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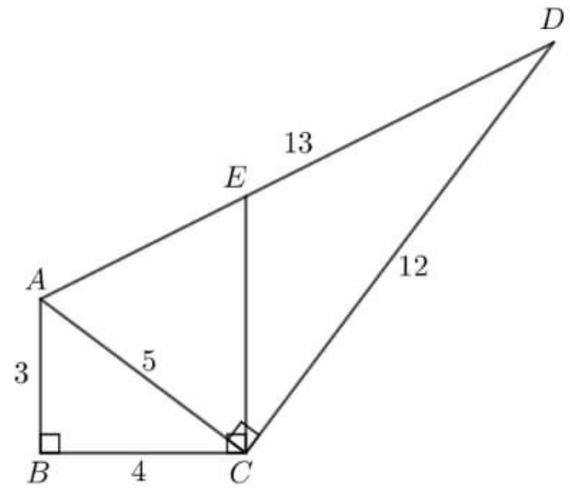
Problem Of The Day 65



ABC and ACD are right-angled triangles with side lengths 3, 4, 5 and 5, 12, 13, respectively, as shown. The line through C at right angles to BC meets AD at the point E.

What is the ratio of the area of the triangle ACE to that of the triangle ECD?

[Give your answer in the form $a : b$, where a and b are positive integers with no common factor other than 1.]

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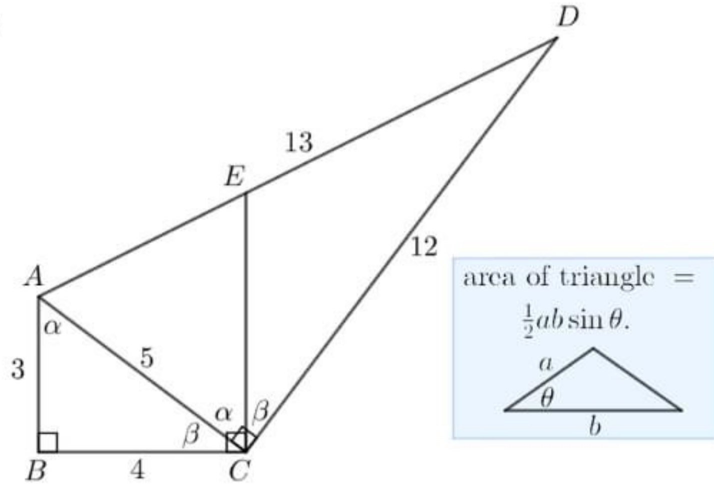
Solution 65

5 : 9



Let $\angle ACE = \alpha$ and $\angle ECD = \beta$.
Because CE is parallel to $BA, \angle BAC = \angle ACE = \alpha$.
Because $\angle ACB + \angle ACE = \angle ECD + \angle ACE = 90^\circ$, it follows that $\angle ACB = \angle ECD = \beta$.

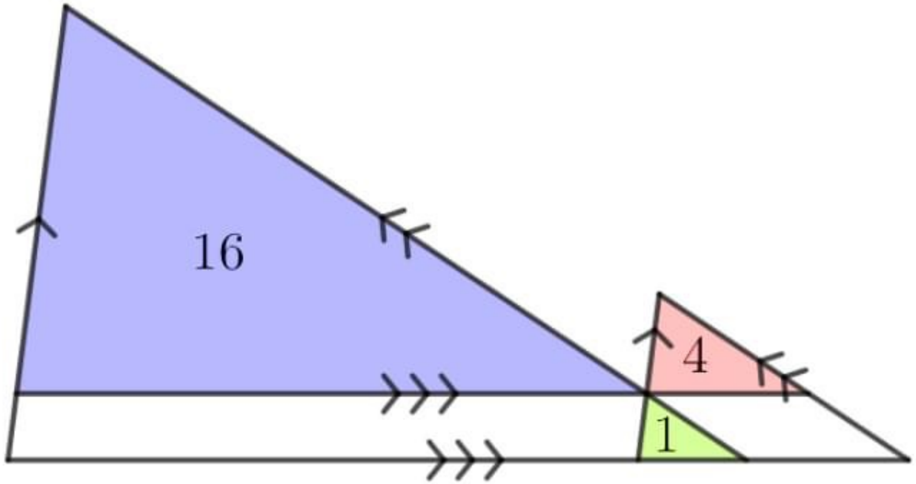
Hence, from the triangle ACB, we have $\sin\alpha = \frac{4}{5}$ and $\sin\beta = \frac{3}{5}$.

Therefore, using the formula $\frac{1}{2}ab\sin\theta$ for the area of a triangle, we have

$$\frac{\text{area } ACE}{\text{area } ECD} = \frac{\frac{1}{2}AC.CE\sin\alpha}{\frac{1}{2}DC.CE\sin\beta} = \frac{AC\sin\alpha}{DC\sin\beta} = \frac{5(\frac{4}{5})}{12(\frac{3}{5})} = \frac{20}{36} = \frac{5}{9}.$$

Hence area ACE : area $DCE = 5:9$.

Problem Of The Day 66



There are three pairs of parallel lines, as indicated, in the above figure.
The three triangles have areas 1, 4 and 16 as shown.

What is the area of the whole shape?

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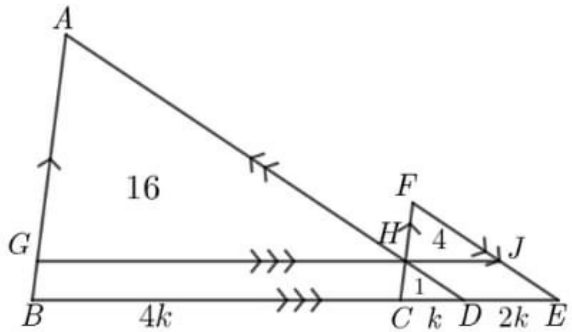
Solution 66

33



Let the points in the figure be labelled as shown. Because of the parallel lines, the triangles HCD, FHJ and AGH are similar. The areas of similar triangles are proportional to the squares of the lengths of corresponding sides. Therefore corresponding sides of the triangles HCD, FHJ and AGH are in the ratio $1:2:4$. We suppose that the lengths of CD, HJ and GH are k, $2k$ and $4k$. Because $HJDE$ and $GHBC$ are parallelograms, $DE = HJ = 2k$ and $BC = GH = 4k$. Therefore $CE = 3k$ and $BD = 5k$.
It follows that the triangle FCE has area 9 and the triangle ABD has area 25. The area of the whole figure is the sum of the areas of these two triangles less the area of the overlap which is the triangle HCD with area 1.
Hence the total area of the figure is $9 + 25 - 1 = 33$.

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Problem Of The Day 67

$$23\times$$

Which positive integers are equal to 23 times the sum of their digits?

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Solution 67

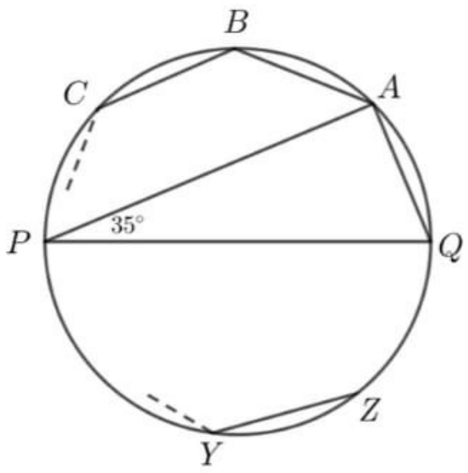
Just 207

It is easy to see that there are no one-digit or two-digit integers that are equal to 23 times the sum of their digits. For example, if n is the two-digit integer "ab", then $n = 10a + b < 23(a + b)$.

Now suppose that $n =$ "abc" is a three-digit integer that equals 23 times the sum of its digits. Then $100a + 10b + c = 23(a + b + c)$. Hence $77a = 13b + 22c$ and so $13b = 77a - 22b$. Therefore b is divisible by 11. Because b is a single digit, $b = 0$. Hence $77a = 22c$ and so $7a = 2c$. The only solution of this equation where a and c are single digits is $a = 2, c = 7$. Hence $n = 207$.

Now if n is a k-digit integer, $n > 10^{k-1}$, whereas the sum of the digits of n is at most $9k$. It may be checked that for $k \geq 4$, we have $23(9k) < 10^{k-1}$. It follows that no integer with four or more digits is equal to 23 times the sum of its digits. Hence 207 is the only solution.

Problem Of The Day 68



The vertices of the triangle PQA lie on a circle, with $\angle QPA = 35°$.
Archie Medes draws the chords AB, BC, and so on, all of the same length as QA.

How many chords does Archie need to draw before he draws a chord YZ, where the point Z coincides with the first point Q?

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Problem Of The Day 69

Mon		2	9	16	23	30
Tue		3	10	17	24	31
Wed		4	11	18	25	
Thu		5	12	19	26	
Fri		6	13	20	27	
Sat		7	14	21	28	
Sun	1	8	15	22	29	

In this leap year, 2020, the 13th of the month is a Friday twice, in March and in November.

What is largest number of Friday the 13^{th}s in any leap year? When does this number next occur?

What is smallest number of Friday the 13^{th}s in any leap year? When does this number next occur?

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Problem Of The Day 70

$$a,\ b,\ c,\ d$$

a, b, c and d are four different integers.

When I take each pair of these integers in turn and add them up, the totals I obtain are 16, 19, 23, 27 and 30, with one of these totals occurring twice.

Find the numbers a, b, c and d.

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